Area on a sphere

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How large is the area F of a region on Earth limited by a geographical longitude increment $\Delta\lambda$ and a latitude increment $\Delta\varphi$?

To answer this question we start from the infinitesimal area to be formally integrated. In a second step an approximation for small area elements is provided.

The infinitesimal area dF can be seen as the product of the infinitesimal edge lengths in East-West direction da and North-South direction db, respectively. Furthermore if the Earth is seen as a sphere it follows that $da = R\cos\varphi d\lambda$ and $db = Rd\varphi$ with the mean Earth radius $R = 6371.23km$.

The Earth surface G is

$$\begin{aligned} G &= \int dF = R^2 \int_{\varphi=-\pi/2}^{\varphi=\pi/2} \int_{\lambda=0}^{\lambda=2\pi} \cos\varphi d\varphi d\lambda = R^2 2\pi \int_{\varphi=-\pi/2}^{\varphi=\pi/2} \cos\varphi d\varphi \\ &= R^2 2\pi \left(\sin\left(\frac{2}{\pi}\right) - \sin\left(\frac{-2}{\pi}\right)\right) = 4\pi R^2 \\ &= 5.101013 * 10^8 km^2. \end{aligned} \tag{1}$$

Generally, the size of an area element on a sphere limited by (λ_1, λ_2) and (φ_1, φ_2) is

$$F = R^2 \int_{\varphi_1}^{\varphi_2} \int_{\lambda_1}^{\lambda_2} \cos\varphi d\varphi d\lambda = R^2 (\lambda_2 - \lambda_1) \left\{\sin(\varphi_2) - \sin(\varphi_1)\right\}. \tag{2}$$

Furthermore it is

$$\sin(\varphi_2) - \sin(\varphi_1) = 2\cos\left(\frac{\varphi_1 + \varphi_2}{2}\right) \sin\left(\frac{\varphi_2 - \varphi_1}{2}\right). \tag{3}$$

$\overline{\varphi} = (\varphi_1 + \varphi_2)/2$ is the mean latitude of the area. The covered latitude is $\Delta\varphi = \varphi_2 - \varphi_1$ and the longitude band is $\Delta\lambda = \lambda_2 - \lambda_1$. Using this eq. (2) rewrites as

$$\boxed{F = 2R^2 \cos(\overline{\varphi}) \Delta\lambda \sin\left(\frac{\Delta\varphi}{2}\right).} \tag{4}$$

For very small latitude bands the argument of the sine becomes very small and the linear approximation of the sine can be used. If the latitude band is $.5^\circ$ the argument of the sine becomes $.5 * \pi/180/2 = 4.36332313\text{E}{-3}$ and the sine itself is $4.36330928\text{E}{-3}$, i.e. the difference is in the range of ppm.

Therefore in the case of small area elements like grid boxes of the size $.5^\circ \times .5^\circ$, it follows that their area can be approximated fairly well by

$$F = R^2 \cos(\overline{\varphi})\Delta\lambda\Delta\varphi. \tag{5}$$

Note that this equation holds true for angles provided in radians. If the angle is expressed in degrees the transformed equation

$$F = R^2 \cos(\overline{\varphi})\Delta\lambda^\circ\Delta\varphi^\circ \frac{\pi^2}{180^2} \tag{6}$$

has to be used, where λ° and φ° is the longitude and latitude in degrees.

If we apply this equation to the Earth we obtain

$$F = 12365.20446 km^2 * \cos(\overline{\varphi})\Delta\lambda^\circ\Delta\varphi^\circ. \tag{7}$$

Finally, in case of $.5^\circ \times .5^\circ$ grid boxes this reduces to

$$F = 3091.301 km^2 * \cos(\overline{\varphi}) \tag{8}$$

which has an error of factor 3.2×10^{-6} (far less then $1km^2$) compared to the correct equation (4)

$$F = 3091.291 km^2 * \cos(\overline{\varphi}). \tag{9}$$